ACADEMIC PRESS

Available online at www.sciencedirect.com

SCIENCE DIRECT

J. Math. Anal. Appl. 276 (2002) 396–421

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Coupled intervals in the discrete calculus of variations: necessity and sufficiency

Roman Hilscher [1,*] and Vera Zeidan [2]

Department of Mathematics, Michigan State University, East Lansing, MI 48824-1027, USA

Received 8 October 2001

Submitted by H.L. Smith

Abstract

In this work we study nonnegativity and positivity of a discrete quadratic functional with separately varying endpoints. We introduce a notion of an interval coupled with 0, and hence, extend the notion of conjugate interval to 0 from the case of fixed to variable endpoint(s). We show that the nonnegativity of the discrete quadratic functional is equivalent to each of the following conditions: The nonexistence of intervals coupled with 0, the existence of a solution to Riccati matrix equation and its boundary conditions. Natural strengthening of each of these conditions yields a characterization of the positivity of the discrete quadratic functional. Since the quadratic functional under consideration could be a second variation of a discrete calculus of variations problem with varying endpoints, we apply our results to obtain necessary and sufficient optimality conditions for such problems. This paper generalizes our recent work in [R. Hilscher, V. Zeidan, Comput. Math. Appl., to appear], where the right endpoint is fixed.

Keywords: Discrete quadratic functional; Coupled interval; Jacobi difference equation; Conjugate interval; Legendre condition; Discrete calculus of variations

* Corresponding author.
E-mail addresses: hilscher@math.msu.edu (R. Hilscher), zeidan@math.msu.edu (V. Zeidan).

1 Research supported by the Czech Grant Agency under grant 201/01/0079.
2 Research supported by the National Science Foundation under grant DMS-0072598.

PII: S0022-247X(02)00495-X

1. Introduction

Let $n, N \in \mathbb{N}$ be given. By the interval $[a, b]$ we always mean the interval of integers $\{a, a+1, \dots, b-1, b\}$. Thus, denote $J := [0, N]$ and $J^* := [0, N+1]$. In this work we study the nonnegativity of the discrete quadratic functional

$$\mathcal{I}(\eta) := \eta_0^T \Gamma_0 \eta_0 + \eta_{N+1}^T \Gamma \eta_{N+1} + \sum_{k=0}^{N} \left\{ \eta_{k+1}^T P_k \eta_{k+1} + 2\eta_{k+1}^T Q_k \Delta\eta_k + \Delta\eta_k^T R_k \Delta\eta_k \right\}$$

over the endpoints constraints

$$M_0 \eta_0 = 0, \qquad M \eta_{N+1} = 0, \tag{1}$$

where $\Gamma_0, \Gamma, P_k, Q_k, R_k$, $k \in J$, are given $n \times n$ matrices, M_0 and M are $r_0 \times n$ and $r \times n$ matrices, respectively, $r_0, r \leqslant n$, $\Gamma_0, \Gamma, P_k, R_k$ are symmetric, $R_k + Q_k^T$ is invertible, and η_k, $k \in J^*$, are n-vectors. The second variation of the following discrete calculus of variations problem with separately varying endpoints

$$\begin{cases} \text{minimize } \mathcal{F}(x) := K_0(x_0) + K(x_{N+1}) + \displaystyle\sum_{k=0}^{N} g(k, x_{k+1}, \Delta x_k), \\ \varphi_0(x_0) = 0, \ \varphi(x_{N+1}) = 0, \end{cases} \tag{P}$$

where $g, K_0, K, \varphi_0, \varphi$ are given twice continuously differentiable functions,

$$g : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \qquad K_0, K : \mathbb{R}^n \to \mathbb{R},$$
$$\varphi_0 : \mathbb{R}^n \to \mathbb{R}^{r_0}, \ r_0 \leqslant n, \qquad \varphi : \mathbb{R}^n \to \mathbb{R}^r, \ r \leqslant n,$$

takes the form $\mathcal{F}_2(\eta) = \frac{1}{2}\mathcal{I}(\eta)$. Hence necessary conditions for the nonnegativity of $\mathcal{I}$ yield necessary optimality conditions for (P), and sufficient conditions for the positivity of $\mathcal{I}$ produce sufficient optimality conditions for (P).

Therefore the focus of this paper is on the study of the characterization of the nonnegativity and positivity of a class of discrete quadratic functionals $\mathcal{J}(\eta)$, see the next section, that includes the functionals $\mathcal{I}(\eta)$ as a special case when the forward differences are expanded. In [17,18] we have thoroughly answered this question for the case when the final endpoint x_{N+1} is fixed, or equivalently, when the variation $\eta_{N+1} = 0$ ($M = I$). The characterization for the fixed final endpoint was given there in terms of a notion of conjugate intervals to 0, the conjoined basis and the focal points notions, and finally in terms of the Riccati equation. The main question for this paper is to deal with the case when both endpoints vary as in (1). To accomplish this goal one may attempt to use the transformation traditionally used to transform the variable endpoints problem into a fixed endpoints problem [10]. This transformation takes a problem of calculus of variations or control into an *optimal control* problem with fixed endpoints. One could then apply to the transformed problem the results available for the fixed endpoints setting [6,8] hoping to obtain the results for the variable endpoints setting after translation. However, this approach would only succeed in characterizing the positivity of the quadratic form in terms of a conjoined basis and the Riccati equation with appropriate boundary conditions. However, since the characterization of the nonnegativity in terms of these concepts is not yet known for the fixed endpoints optimal control problem, then this approach cannot be employed to characterize in terms of these concepts the nonnegativity of the quadratic form in the calculus of variations setting. Furthermore, the nature of this transformation does not allow the possibility of characterizing either the positivity or the nonnegativity of the quadratic form in terms of a "coupled interval" notion. Therefore, it is imperative that we use instead a direct approach.

The task we undertake here will definitely require developing a notion of "coupled intervals" with 0, which extends the notion of conjugate intervals known for the fixed final endpoint case. While this is an open problem in the discrete setting, the coupled points theory was introduced for the continuous case by Zeidan and Zezza [24] and extensively used in characterizing the continuous quadratic functionals with variable endpoints, see, e.g., [11,12,20–23,25]. For instance, it is known [25] that under the strengthened Legendre condition the continuous quadratic functional $\mathcal{G}_2$ is nonnegative over the endpoints constraints of the type (1) if and only if $\mathcal{G}_2$ is "regular" and no point in $[0, T)$ is coupled with 0. In [23] a necessary condition for the nonnegativity of $\mathcal{G}_2$ is obtained in terms of a conjoined basis and another one in terms of the Riccati equation with appropriate boundary conditions. For the positivity of $\mathcal{G}_2$ over constraints of type (1), it is shown in [22] and [25] that under the same strengthened Legendre condition, $\mathcal{G}_2 > 0$ if and only if $\mathcal{G}_2$ is "regular" and no point in $[0, T]$ is coupled with 0. This latter is equivalent in [22] to a condition in terms of a conjoined basis, and to another one in terms of a Riccati equation with appropriate boundary conditions.

In this paper a thorough study is presented of the nonnegativity and positivity of the discrete quadratic functional of the form $\mathcal{J}(\eta)$. First in Section 2 we list auxiliary results that are vital for the main results of the paper. That includes the results obtained in [18] for the case when the final endpoint is fixed. In Section 3 the discrete (strengthened) Legendre condition is obtained. Then, the notion of coupled intervals with 0 is introduced based on the idea of continuing a solution of the Euler–Lagrange equation associated with $\mathcal{J}(\eta)$ by a constant value α in the kernel of M. As opposed to the continuous time case, in the discrete setting the choice of α is not unique because η is not continuous. Furthermore, the discreteness of the time interval allows the value of the partial sum $\mathcal{J}_m$, defined in Section 2, to jump from positive to negative. Thus, these differences are reflected in the definition of coupled intervals with 0 (see Definition 2) by having a random extension in the kernel of M and by having $d_m \leqslant 0$ and not just $d_m = 0$, which means that no transversality condition at $(m, m+1]$ is required. Theorem 2 is a characterization of the nonnegativity of $\mathcal{J}$ and generalizes the corresponding result in [18] to the case of varying endpoints. Note that the condition that $(N, N+1]$ is not "strictly coupled" with 0 is needed in our discrete setting while in the continuous case it is not required, since a time-limiting argument there was possible. As in the continuous case, the "regularity" of $\mathcal{J}(\eta)$ is needed. We show that this regularity condition is equivalent to $(0,1]$ not being strictly coupled with 0. Theorem 3 of Section 3 expresses the equivalence of the nonnegativity of $\mathcal{J}$ with the positivity of the partial sum $\mathcal{J}_m$ obtained by taking η to be a constant value α on $[m+1, N+1]$. Theorem 4 is a characterization of the positivity of $\mathcal{J}$ in terms of the coupled intervals. In the special case when $\mathcal{J}$ corresponds to $\mathcal{I}$ the characterization of $\mathcal{I} > 0$ in terms of a conjoined basis condition is known in [7, Theorem 2] and [10, Theorem 3]. In Section 4 we specialize $\mathcal{J}$ to be the functional $\mathcal{I}$ and we show in this case that the nonnegativity and positivity of $\mathcal{I}$ can be expressed in terms of the explicit Riccati difference equation and its boundary conditions. In Section 5 necessary and sufficient optimality conditions for (P) are derived in terms of either of the equivalent conditions obtained in Sections 3 and 4.

2. Auxiliary results

Let us introduce the terminology and notation used throughout the paper. By $\operatorname{Ker} A$, $\operatorname{Im} A$, A^T, A^{T-1}, $A^\dagger$, $A \geqslant 0$, and $A > 0$ we denote the kernel, image, transpose, inverse of the transpose, Moore–Penrose generalized inverse, positive semidefiniteness, and positive definiteness, respectively, of the matrix A. The forward difference operator is denoted by Δ, i.e., $\Delta y_k = y_{k+1} - y_k$. We adopt the convention that $\sum_{k=a}^{b} u_k = 0$ when $a > b$. We start with basic properties of solutions of the *three term recurrence equation*

$$-S_{k+1}\eta_{k+2} + T_{k+1}\eta_{k+1} - S_k^T \eta_k = 0, \quad k \in [0, N-1], \tag{$\mathcal{T}$}$$

where $S, T : J \to \mathbb{R}^{n\times n}$, T_k is symmetric, and S_k is invertible. The Jacobi difference equation for a discrete variational problem (P) is a three term recurrence of the form $(\mathcal{T})$, as we shall see in Section 4. For more details see [2,5,18]. As usual, the vector solutions of $(\mathcal{T})$ will be denoted by small letters, and the $n \times n$ matrix solutions by capital ones. For any two solutions X, Y of $(\mathcal{T})$ the *Wronskian matrix* $\{X, Y\} := X_k^T S_k Y_{k+1} - X_{k+1}^T S_k^T Y_k$ is constant. A solution X of $(\mathcal{T})$ is a *conjoined basis* if $\operatorname{rank}\binom{X_0}{X_1} = n$ and $\{X, X\} = 0$, i.e., $X_k^T S_k X_{k+1}$ is symmetric. A solution η of $(\mathcal{T})$ is said to have a *generalized zero* in the interval $(m, m+1]$ if

$$\eta_m \neq 0 \quad \text{and} \quad d_m := \eta_m^T S_m \eta_{m+1} \leqslant 0. \tag{2}$$

When the left endpoint is varying and the right endpoint is fixed, the authors introduced in [17,18] a notion of an interval conjugate to 0. Let $m \in J$. An interval $(m, m+1]$ is *conjugate to* 0 if there exists a solution η of $(\mathcal{T})$ that has a generalized zero in $(m, m+1]$ and satisfies, for some vector $\gamma_0 \in \mathbb{R}^{r_0}$, the initial boundary and transversality conditions

$$M_0\eta_0 = 0, \quad \eta_1 = S_0^{-1}\bigl(T_0\eta_0 + M_0^T\gamma_0\bigr). \tag{3}$$

If $(m, m+1]$ is conjugate to 0 with an associated η that satisfies the strict inequality in (2), then we say that the interval $(m, m+1]$ is *strictly conjugate to* 0.

Consider the quadratic functional $\mathcal{J}$ defined by

$$\mathcal{J}(\eta) := \eta_0^T T_0\eta_0 + \sum_{k=0}^{N}\bigl\{\eta_{k+1}^T T_{k+1}\eta_{k+1} - \eta_k^T S_k\eta_{k+1} - \eta_{k+1}^T S_k^T\eta_k\bigr\},$$

subject to the sequences $\{\eta_k\}_{k=0}^{N+1}$ satisfying the boundary conditions (1). Such η is called *admissible*. We say that $\mathcal{J}$ is *nonnegative* ($\mathcal{J} \geqslant 0$) if $\mathcal{J}(\eta) \geqslant 0$ for all admissible η. The functional $\mathcal{J}$ is *positive definite* ($\mathcal{J} > 0$) if $\mathcal{J}(\eta) > 0$ for all admissible η, $\eta \not\equiv 0$.

This paper extends the following result from fixed to varying right endpoint. The matrix $\mathcal{M}_0$ in Proposition 1 below is the projection defined by

$$\mathcal{M}_0 := M_0^T\bigl(M_0M_0^T\bigr)^{-1}M_0, \tag{4}$$

and Y_0 is the $n \times r_0$ matrix whose columns form an orthonormal basis for $\operatorname{Ker} M_0$. Let $\mathcal{L}_k$, $k \in J$, be the block tridiagonal matrices associated with $\mathcal{J}$. More precisely, let $\mathcal{L}_0 := Y_0^T T_0 Y_0$, and for $k \in [1, N]$

$$\mathcal{L}_k := \begin{pmatrix} Y_0^T T_0 Y_0 & -Y_0^T S_0 & 0 & \dots & 0 \\ -S_0^T Y_0 & T_1 & -S_1 & \ddots & \vdots \\ 0 & -S_1^T & T_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -S_{k-1} \\ 0 & \dots & 0 & -S_{k-1}^T & T_k \end{pmatrix}.$$

If $M_0 = I$, then $Y_0 = 0$ and hence, delete the first row and column in the above matrices and start just with $\mathcal{L}_1 = T_1$.

Proposition 1 (Right endpoint fixed, [18, Theorem 2]). *Suppose that S_k, $k \in J$, are invertible, M_0 has full rank, and $M = I$. Then the following conditions are equivalent.*

(i) $\mathcal{J} \geqslant 0$, *i.e., $\mathcal{J}(\eta) \geqslant 0$ for all sequences $\{\eta_k\}_{k=0}^{N+1}$ with $M_0\eta_0 = 0$, $\eta_{N+1} = 0$, and the initial strengthened Legendre condition $Y_0^T T_0 Y_0 > 0$ holds if $M_0 \neq 0$.*

(ii) $\mathcal{L}_k > 0$ *for all* $k \in [0, N-1]$, *and* $\mathcal{L}_N \geqslant 0$.

(iii) *There is no interval $(m, m+1] \subseteq (0, N]$ conjugate to* 0, *and $(N, N+1]$ is not strictly conjugate to* 0, *i.e., the Jacobi necessary condition holds.*

(iv) *The solution X of $(\mathcal{T})$ with*

$$X_0 = I - \mathcal{M}_0, \qquad X_1 = S_0^{-1}(T_0 X_0 + \mathcal{M}_0), \tag{5}$$

has X_k invertible for all $k \in [1, N]$ and satisfies

$$\begin{aligned} &Y_0^T X_0^T S_0 X_1 Y_0 > 0, \\ &X_k^T S_k X_{k+1} > 0 \quad \textit{for all } k \in [1, N-1], \\ &X_N^T S_N X_{N+1} \geqslant 0. \end{aligned}$$

(v) *The solution X of $(\mathcal{T})$ with* (5) *has X_k invertible for all $k \in [1, N]$ and the matrices D_k defined by*

$$D_k := X_k X_{k+1}^{-1} S_k^{-1}, \quad k \in [0, N-1],$$

satisfy

$$\begin{aligned} &D_0 > 0 \quad \textit{on } \operatorname{Im} T_0 Y_0, \\ &D_k > 0 \quad \textit{for all } k \in [1, N-1], \\ &T_N - S_{N-1}^T D_{N-1} S_{N-1} \geqslant 0. \end{aligned}$$

(vi) *The matrices H_k defined recursively by*

$$\begin{aligned} &H_0 := Y_0^T T_0 Y_0, \\ &H_1 := T_1 - S_0^T Y_0 H_0^{-1} Y_0^T S_0, \\ &H_{k+1} := T_{k+1} - S_k^T H_k^{-1} S_k, \quad k \in [1, N-1], \end{aligned}$$

satisfy

$$H_k > 0 \quad \textit{for all } k \in [0, N-1], \quad H_N \geqslant 0.$$

We will also need more general treatment of linear difference equations, namely the concept of *linear Hamiltonian difference system*

$$\Delta\eta_k = A_k \eta_{k+1} + B_k q_k, \qquad \Delta q_k = C_k \eta_{k+1} - A_k^T q_k, \tag{$\mathcal{H}$}$$

where $A, B, C : J \to \mathbb{R}^{n\times n}$, B_k, C_k are symmetric, and $I - A_k$ is invertible; we denote $\tilde{A}_k := (I - A_k)^{-1}$. For a more detailed discussion we refer to [1,5,6,13, 14]. Again, the vector solutions of $(\mathcal{H})$ will be denoted by small letters, and the $n \times n$ matrix solutions by capital ones. A pair (η, q) is said to be *admissible* (on J^*) if it satisfies the first equation of $(\mathcal{H})$, i.e., $\Delta\eta_k = A_k\eta_{k+1} + B_k q_k$, $k \in J$, and the boundary conditions (1). Let (X, U), $(\widetilde{X}, \widetilde{U})$ be solutions of $(\mathcal{H})$. Then $X_k^T \widetilde{U}_k - U_k^T \widetilde{X}_k \equiv W$, where W is a constant $n \times n$ matrix, sometimes called a *Wronskian* of the solutions (X, U) and $(\widetilde{X}, \widetilde{U})$. If $W = I$, then these solutions are called *normalized*. A solution (X, U) is said to be a *conjoined basis* if $X^T U$ is symmetric and $\operatorname{rank}\binom{X}{U} = n$.

Following [6], a solution (X, U) of $(\mathcal{H})$ is said to have *no focal points in* $(0, N+1]$, provided

$$\operatorname{Ker} X_{k+1} \subseteq \operatorname{Ker} X_k \quad \text{and} \quad D_k := X_k X_{k+1}^{\dagger} \tilde{A}_k B_k \geqslant 0$$

holds for all $k \in J$. Observe that D_k's are symmetric when the kernel condition holds [6, Lemma 2]. System $(\mathcal{H})$ is said to be *disconjugate on* J^* if the principal solution at 0 has no focal points in $(0, N+1]$. The interval $(m, m+1]$ is a *generalized zero* of a solution (η, q) of $(\mathcal{H})$, provided

$$\eta_m \neq 0, \quad \eta_{m+1} \in \operatorname{Im} \tilde{A}_m B_m \quad \text{and} \quad \eta_m^T B_m^{\dagger} (I - A_m) \eta_{m+1} \leqslant 0. \tag{6}$$

Associated with $(\mathcal{H})$ is the *Riccati matrix difference equation*

$$R[W]_k := \Delta W_k - C_k + A_k^T W_k + (W_{k+1} - C_k) \tilde{A}_k (A_k + B_k W_k) = 0, \quad (\mathcal{R})$$

and the quadratic functional

$$\mathcal{K}(\eta, q) := \eta_0^T \Gamma_0 \eta_0 + \eta_{N+1}^T \Gamma \eta_{N+1} + \sum_{k=0}^{N} \left\{ \eta_{k+1}^T C_k \eta_{k+1} + q_k^T B_k q_k \right\}.$$

Remark 1. When the three term recurrence equation $(\mathcal{T})$ is the Euler–Lagrange equation of the quadratic functional $\mathcal{I}$ with R_k and $R_k + Q_k^T$ invertible, then $(\mathcal{T})$ can be rewritten as the linear Hamiltonian system $(\mathcal{H})$, see Section 4. In this case the corresponding notions for $(\mathcal{T})$ and $(\mathcal{H})$, such as generalized zeros, conjugate intervals, quadratic functionals $\mathcal{I}$, $\mathcal{J}$, and $\mathcal{K}$, etc., coincide. It should be clear from the context which of the above notions we refer to.

3. Nonnegativity and positivity of $\mathcal{J}$

In this section we derive the (strengthened) Legendre condition for the quadratic functional $\mathcal{J}$ with boundary conditions (1) which corresponds to the three term recurrence equation $(\mathcal{T})$ defined in the previous section. We also introduce the notion of coupled intervals with 0. We show in Theorem 2 that in the case of separately varying endpoints, the coupled intervals notion plays here the same role of conjugate intervals in Proposition 1.

Thus, let $T_k, S_k \in \mathbb{R}^{n \times n}$, $M_0 \in \mathbb{R}^{r_0 \times n}$, $r_0 \leqslant n$, $M \in \mathbb{R}^{r \times n}$, $r \leqslant n$, with T_k symmetric, $k \in J^*$. Let Y_0 and Y be the $n \times r_0$ and $n \times r$ matrices whose columns form orthonormal bases for $\operatorname{Ker} M_0$ and $\operatorname{Ker} M$, respectively, i.e., $\operatorname{Ker} M_0 = \operatorname{Im} Y_0$ and $\operatorname{Ker} M = \operatorname{Im} Y$.

In order to ensure uniqueness and continuation of solutions of $(\mathcal{T})$, our general assumption is

$$S_k \text{ invertible for all } k \in J \text{ and } M_0, M \text{ have full rank.} \tag{A1}$$

Remark 2. The following is a complement to our notation introduced in (4).

(i) Denote by $\mathcal{M}$ the projection

$$\mathcal{M} := M^T\left(MM^T\right)^{-1}M. \tag{7}$$

It follows easily that $M\alpha = 0$ iff $\mathcal{M}\alpha = 0$, and similarly for $\mathcal{M}_0$, the projection defined by (4).

(ii) The matrices Y_0 and Y also define projections $\mathcal{Y}_0 := Y_0Y_0^T \in \mathbb{R}^{n\times n}$ and $\mathcal{Y} := YY^T \in \mathbb{R}^{n\times n}$ (observe $Y_0^TY_0 = I_{r_0\times r_0}$ and $Y^TY = I_{r\times r}$). One can easily verify that

$$\mathcal{Y}_0 = I - \mathcal{M}_0, \qquad \mathcal{Y} = I - \mathcal{M}. \tag{8}$$

Due to the fact that the right endpoint of η is varying, we shall define in addition to the matrices $\mathcal{L}_k$, $k \in J$, the matrix

$$\mathcal{L}_{N+1} := \begin{pmatrix} Y_0^TT_0Y_0 & -Y_0^TS_0 & \dots & 0 & 0 \\ -S_0^TY_0 & T_1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & -S_{N-1} & 0 \\ 0 & \dots & -S_{N-1}^T & T_N & -S_NY \\ 0 & \dots & 0 & -Y^TS_N^T & Y^TT_{N+1}Y \end{pmatrix},$$

which is the block tridiagonal matrix representation of the quadratic functional $\mathcal{J}$. If $M = I$, then $Y = 0$ and delete the last row and column of $\mathcal{L}_{N+1}$. And similarly as before, if $M_0 = I$, then $Y_0 = 0$ and delete the first row and column of $\mathcal{L}_{N+1}$. For the case when both boundary conditions are zero see [2] and also [9,15,16]. Now, if η is admissible for $\mathcal{J}$, then $\eta_0 = Y_0\alpha$ and $\eta_{N+1} = Y\beta$ for some $\alpha \in \mathbb{R}^{r_0}$ and $\beta \in \mathbb{R}^r$. With the notation $\eta^* := (\alpha^T\ \eta_1^T\ \dots\ \eta_N^T\ \beta^T)^T$ we have readily the following lemma.

Lemma 1. *If η is admissible for $\mathcal{J}$, then $\mathcal{J}(\eta) = (\eta^*)^T\mathcal{L}_{N+1}\eta^*$.*

As we shall see next, it is possible to also characterize $\mathcal{J} \geqslant 0$ via certain partial quadratic functionals $\mathcal{J}_m$, when the varying right endpoint is moved to the point m. This endpoint transfer yields a cumulative final cost in the functional $\mathcal{J}_m$.

For $m \in J$ and $\alpha \in \mathbb{R}^n$ we define the partial quadratic functional

$$\begin{aligned}\mathcal{J}_m(\eta,\alpha) := {}& \eta_0^TT_0\eta_0 - \eta_m^TS_m\alpha - \alpha^TS_m^T\eta_m + \alpha^T\widetilde{T}_{m+1}\alpha \\ & + \sum_{k=0}^{m-1}\left\{\eta_{k+1}^TT_{k+1}\eta_{k+1} - \eta_k^TS_k\eta_{k+1} - \eta_{k+1}^TS_k^T\eta_k\right\},\end{aligned}$$

where the cumulative final cost coefficient $\widetilde{T}_{m+1}$ is given by

$$\widetilde{T}_{m+1} := T_{N+1} + \sum_{k=m+1}^{N}\left(T_k - S_k - S_k^T\right), \tag{9}$$

and where a pair $\eta = \{\eta_k\}_{k=0}^{m}$ and α is *admissible for* $\mathcal{J}_m$ provided

$$M_0\eta_0 = 0, \qquad M\alpha = 0. \tag{10}$$

Note that $\widetilde{T}_{N+1} = T_{N+1}$. Functional $\mathcal{J}_m$ is *nonnegative* ($\mathcal{J}_m \geqslant 0$) if $\mathcal{J}_m(\eta,\alpha) \geqslant 0$ for any admissible η and α. Functional $\mathcal{J}_m$ is *positive definite* ($\mathcal{J}_m > 0$) if $\mathcal{J}_m(\eta,\alpha) > 0$ for any admissible η and α, not both zero. With $\eta_{N+1} := \alpha$ we have then $\mathcal{J} = \mathcal{J}_N$.

Remark 3. When $M = I$, these partial quadratic functionals are represented by the matrices $\mathcal{L}_k$. Thus, if the right endpoint is fixed, the role of $\mathcal{J}_m$ is hidden already in (ii) of Proposition 1, and in fact the functionals $\mathcal{J}_m$ are not needed in this case.

Lemma 2 (Characterization of $\mathcal{J} \geqslant 0$). *Assume that* (A1) *holds. Then*

$$\mathcal{J} \geqslant 0 \ \Leftrightarrow\ \mathcal{J}_m \geqslant 0 \textit{ for all } m \in J.$$

Proof. "$\Leftarrow$" The nonnegativity of $\mathcal{J}$ follows by taking $m = N$.

"$\Rightarrow$" Suppose $\mathcal{J} \geqslant 0$, fix $m \in J$, and let $\{\eta_k\}_{k=0}^{m}$ and α be admissible for $\mathcal{J}_m$. Define a sequence $\{\tilde\eta_k\}_{k=0}^{N+1}$ by

$$\tilde\eta_k := \begin{cases} \eta_k & \text{for } k \in [0,m], \\ \alpha & \text{for } k \in [m+1, N+1]. \end{cases} \tag{11}$$

Then $M_0\tilde\eta_0 = M_0\eta_0 = 0$ and $M\tilde\eta_{N+1} = M\alpha = 0$, i.e., $\tilde\eta$ is admissible for $\mathcal{J}$. It follows that $\mathcal{J}_m(\eta) = \mathcal{J}(\tilde\eta) \geqslant 0$, so that $\mathcal{J}_m \geqslant 0$. □

Theorem 1 (Discrete (strengthened) Legendre condition). *If* $\mathcal{J} \geqslant 0$, *then the discrete Legendre condition*

$$Y_0^T T_0 Y_0 \geqslant 0, \quad T_k \geqslant 0 \quad \textit{for all } k \in [1,N], \quad \textit{and} \tag{12}$$

$$Y^T \widetilde{T}_k Y \geqslant 0 \quad \textit{for all } k \in [1, N+1] \tag{13}$$

holds. If $\mathcal{J} > 0$, *then the discrete strengthened Legendre condition*

$$Y_0^T T_0 Y_0 > 0, \quad T_k > 0 \quad \textit{for all } k \in [1,N], \quad \textit{and} \tag{14}$$

$$Y^T \widetilde{T}_k Y > 0 \quad \textit{for all } k \in [1, N+1] \tag{15}$$

holds.

Proof. Let $m \in J$. Choose an admissible η with all entries zero except at $k = m$, say $\eta_m = c$. Then $\mathcal{J}(\eta) = c^T T_m c$ and (12), (14) follow. If we take η to be $\eta_k = 0$ on $[0,m]$, and $\eta_k = \alpha$ on $[m+1, N+1]$ for some $\alpha \in \operatorname{Ker} M$, then such η is admissible and $\mathcal{J}(\eta) = \alpha^T \widetilde{T}_{m+1}\alpha$. Thus, conditions (13) and (15) follow as well. □

Definition 1 (*Regularity*). We say that $\mathcal{J}$ is *regular* if for any $\eta_0, \alpha \in \mathbb{R}^n$ with $M_0\eta_0 = 0$, $M\alpha = 0$, we have $\mathcal{J}_0(\eta_0, \alpha) \geqslant 0$, where

$$\mathcal{J}_0(\eta_0, \alpha) := \eta_0^T T_0 \eta_0 - \eta_0^T S_0 \alpha - \alpha^T S_0^T \eta_0 + \alpha^T \widetilde{T}_1 \alpha.$$

If in addition $\mathcal{J}_0(\eta_0, \alpha) > 0$ for $(\eta_0, \alpha) \neq 0$, then $\mathcal{J}$ is said to be *strongly regular*.

Remark 4. If $\mathcal{J} \geqslant 0$ (resp. $\mathcal{J} > 0$), then $\mathcal{J}$ is regular (resp. strongly regular).

The following is a notion of an interval coupled with 0, which is a natural extension of the conjugate interval to 0 from fixed to variable right endpoint.

Definition 2 (*Coupled interval*). Let $m \in J$. An interval $(m, m+1]$ is *coupled with* 0 if there exists a solution η of ($\mathcal{T}$) satisfying the initial boundary and transversality conditions (3), $\eta_m \neq 0$, and for some $\alpha \in \operatorname{Ker} M$, $\eta_k \not\equiv \alpha$ on $[m+1, N+1]$ (drop if $m = N$), and

$$\begin{aligned} d_m(\eta_m, \eta_{m+1}, \alpha) := \eta_m^T S_m \eta_{m+1} - \eta_m^T S_m \alpha - \alpha^T S_m^T \eta_m \\ + \alpha^T \widetilde{T}_{m+1} \alpha \leqslant 0. \end{aligned} \tag{16}$$

If $(m, m+1]$ is coupled with 0 and the above inequality is strict, we say that the interval $(m, m+1]$ is *strictly coupled with* 0.

Remark 5. Observe that any interval conjugate to 0 is also coupled with 0, since we may take $\alpha = 0$. Also, if $M_0 = I$, then $\eta_0 = 0$ and hence the interval $(0, 1]$ cannot be coupled with 0. Obviously, in general $(m, m+1]$ could be coupled with 0 and not necessarily conjugate to 0. In this case $\alpha \neq 0$.

Lemma 3. *Let $m \in J$ and $\alpha \in \operatorname{Ker} M$. Suppose that $\{\eta_k\}_{k=0}^{N+1}$ is a solution of* ($\mathcal{T}$) *satisfying the initial boundary and transversality conditions* (3). *Then $\tilde{\eta}$ defined by* (11) *is admissible and $\mathcal{J}(\tilde{\eta}) = \mathcal{J}_m(\eta, \alpha) = d_m(\eta_m, \eta_{m+1}, \alpha)$, where d_m is defined by* (16).

Proof. For $\{\eta_k\}_{k=0}^{N+1}$ and $\{\tilde{\eta}_k\}_{k=0}^{N+1}$ defined above we have

$$\begin{aligned} \mathcal{J}(\tilde{\eta}) &= \tilde{\eta}_0^T T_0 \tilde{\eta}_0 + \sum_{k=0}^{N} \big\{ \tilde{\eta}_{k+1}^T T_{k+1} \tilde{\eta}_{k+1} - \tilde{\eta}_k^T S_k \tilde{\eta}_{k+1} - \tilde{\eta}_{k+1}^T S_k^T \tilde{\eta}_k \big\} \\ &= \tilde{\eta}_0^T \big(T_0 \tilde{\eta}_0 - S_0 \tilde{\eta}_1 \big) + \sum_{k=0}^{N-1} \tilde{\eta}_{k+1}^T \big\{ -S_{k+1} \tilde{\eta}_{k+2} + T_{k+1} \tilde{\eta}_{k+1} - S_k^T \tilde{\eta}_k \big\} \\ &\quad + \tilde{\eta}_{N+1}^T \big(T_{N+1} \tilde{\eta}_{N+1} - S_N^T \tilde{\eta}_N \big). \end{aligned}$$

We distinguish three cases: (a) $m \in [1, N-1]$, (b) $m = N$, and (c) $m = 0$.

(a) *Case* $m \in [1, N-1]$. Since $\tilde{\eta}_k = \eta_k$ on $[0, m]$, using (3) we get from the above computation

$$\begin{aligned}
\mathcal{J}(\tilde{\eta}) &= \eta_0^T (T_0\eta_0 - S_0\eta_1) \\
&\quad + \sum_{k=0}^{m-2} \eta_{k+1}^T \{-S_{k+1}\eta_{k+2} + T_{k+1}\eta_{k+1} - S_k^T \eta_k\} \\
&\quad + \eta_m^T \big(-S_m\alpha + T_m\eta_m - S_{m-1}^T\eta_{m-1}\big) \\
&\quad + \alpha^T \big(-S_{m+1}\alpha + T_{m+1}\alpha - S_m^T\eta_m\big) \\
&\quad + \alpha^T \sum_{k=m+1}^{N-1} \big(-S_{k+1} + T_{k+1} - S_k^T\big)\alpha + \alpha^T \big(T_{N+1} - S_N^T\big)\alpha \\
&= \eta_m^T \big(T_m\eta_m - S_{m-1}^T\eta_{m-1}\big) - \eta_m^T S_m\alpha - \alpha^T S_m^T \eta_m \\
&\quad + \alpha^T \left\{ -\sum_{k=m+1}^{N} S_k - \sum_{k=m+1}^{N} S_k^T + \sum_{k=m+1}^{N+1} T_k \right\} \alpha = d_m.
\end{aligned}$$

(b) *Case* $m = N$. Similarly as in part (a) we have

$$\begin{aligned}
\mathcal{J}(\tilde{\eta}) &= \eta_0^T (T_0\eta_0 - S_0\eta_1) \\
&\quad + \sum_{k=0}^{N-2} \eta_{k+1}^T \{-S_{k+1}\eta_{k+2} + T_{k+1}\eta_{k+1} - S_k^T \eta_k\} \\
&\quad + \eta_N^T \big(-S_N\alpha + T_N\eta_N - S_{N-1}^T\eta_{N-1}\big) + \alpha^T \big(T_{N+1}\alpha - S_N^T\eta_N\big) \\
&= \eta_N^T \big(T_N\eta_N - S_{N-1}^T\eta_{N-1}\big) \\
&\quad - \eta_N^T S_N\alpha - \alpha^T S_N^T \eta_N + \alpha^T T_{N+1}\alpha = d_N.
\end{aligned}$$

(c) *Case* $m = 0$. From the transversality condition in (3) we obtain $\eta_0^T S_0\eta_1 = \eta_0^T T_0\eta_0$, so that

$$\begin{aligned}
\mathcal{J}(\tilde{\eta}) &= \eta_0^T (T_0\eta_0 - S_0\alpha) + \alpha^T \big(-S_1\alpha + T_1\alpha - S_0^T\eta_0\big) \\
&\quad + \alpha^T \sum_{k=1}^{N-1} \big(-S_{k+1} + T_{k+1} - S_k^T\big)\alpha + \alpha^T \big(T_{N+1}\alpha - S_N^T\alpha\big) \\
&= \eta_0^T S_0\eta_1 - \eta_0^T S_0\alpha - \alpha^T S_0^T \eta_0 \\
&\quad + \alpha^T \left\{ -\sum_{k=1}^{N} S_k - \sum_{k=1}^{N} S_k^T + \sum_{k=1}^{N+1} T_k \right\} \alpha = d_0.
\end{aligned}$$

In all three cases we showed that $\mathcal{J}(\tilde{\eta}) = d_m$ and the proof is complete. $\square$

One of the main contributions of this paper is the following theorem. As in Proposition 1, we need only the strengthened Legendre condition at $k = 0$:

$$Y_0^T T_0 Y_0 > 0 \quad \text{if } M_0 \neq I. \tag{17}$$

Theorem 2 (Characterization of $\mathcal{J} \geqslant 0$). *Suppose that* (A1) *holds. Then the following conditions are equivalent.*

(i) $\mathcal{J} \geqslant 0$, *i.e.,* $\mathcal{J}(\eta) \geqslant 0$ *for all admissible* η, *and condition* (17) *holds.*
(ii) $\mathcal{L}_k > 0$ *for all* $k \in [0, N-1]$, $\mathcal{L}_N \geqslant 0$, *and* $\mathcal{L}_{N+1} \geqslant 0$.
(iii) $\mathcal{J}$ *is regular, there is no interval* $(m, m+1] \subseteq (1, N]$ *coupled with* 0, *interval* $(N, N+1]$ *is not strictly coupled with* 0, *and* $(0, 1]$ *is not conjugate to* 0.
(iv) *The solution* X *of* $(\mathcal{T})$ *defined as in* (5), *i.e.,*

$$X_0 = I - \mathcal{M}_0, \quad X_1 = S_0^{-1}(T_0 X_0 + \mathcal{M}_0),$$

has X_k *invertible for all* $k \in [1, N]$ *and satisfies*

$$Y_0^T X_0^T S_0 X_1 Y_0 > 0, \tag{18}$$

$$X_k^T S_k X_{k+1} > 0 \quad \textit{for all } k \in [1, N-1], \tag{19}$$

$$\begin{pmatrix} X_N^T S_N X_{N+1} & -X_N^T S_N Y \\ -Y^T S_N^T X_N & Y^T T_{N+1} Y \end{pmatrix} \geqslant 0. \tag{20}$$

(v) *The solution* X *of* $(\mathcal{T})$ *with* (5) *has* X_k *invertible for all* $k \in [1, N]$ *and the matrices* D_k *defined by*

$$D_k := X_k X_{k+1}^{-1} S_k^{-1}, \quad k \in [0, N-1],$$

satisfy

$$D_0 > 0 \quad \textit{on } \operatorname{Im} T_0 Y_0, \tag{21}$$

$$D_k > 0 \quad \textit{for all } k \in [1, N-1], \tag{22}$$

$$\begin{pmatrix} T_N - S_{N-1}^T D_{N-1} S_{N-1} & -S_N Y \\ -Y^T S_N^T & Y^T T_{N+1} Y \end{pmatrix} \geqslant 0. \tag{23}$$

(vi) *The matrices* H_k *defined recursively by*

$$H_0 := Y_0^T T_0 Y_0, \tag{24}$$

$$H_1 := T_1 - S_0^T Y_0 H_0^{-1} Y_0^T S_0, \tag{25}$$

$$H_{k+1} := T_{k+1} - S_k^T H_k^{-1} S_k, \quad k \in [1, N-1], \tag{26}$$

satisfy

$$H_k > 0 \quad \textit{for all } k \in [0, N-1], \quad \textit{and}$$
$$\begin{pmatrix} H_N & -S_N Y \\ -Y^T S_N^T & Y^T T_{N+1} Y \end{pmatrix} \geqslant 0. \tag{27}$$

Remark 6. In the case where $M = I$, the nonnegativity of $\mathcal{J}$ and (17) imply by (i) $\Rightarrow$ (ii) in Proposition 1 that the strengthened Legendre condition (14) holds, i.e., $T_k > 0$ for all $k \in [1, N]$. However, when $M \neq I$, a consequence of (i) $\Rightarrow$ (ii) in Theorem 2 is that $\mathcal{J} \geqslant 0$ and (17) imply (14) and $Y^T T_{N+1} Y \geqslant 0$ only, which is not the entire strengthened Legendre condition (14), (15).

Proof of Theorem 2. We proceed by showing the following steps

$$\text{(ii)} \;\Rightarrow\; \text{(i)} \;\Rightarrow\; \text{(iii)} \;\Rightarrow\; \text{(iv)} \;\Rightarrow\; \text{(v)} \;\Rightarrow\; \text{(vi)} \;\Rightarrow\; \text{(ii)}.$$

"(ii) $\Rightarrow$ (i)" This is a direct consequence of Lemma 1.

"(i) $\Rightarrow$ (iii)" Regularity of $\mathcal{J}$ follows from Remark 4. Let $m \in J$ and suppose that $(m, m+1]$ is coupled with 0. Let η be the solution of $(\mathcal{T})$ from the definition of coupled interval with the corresponding α. Recall that $\eta_k \not\equiv \alpha$ on $[m+1, N+1]$. Define an admissible $\tilde{\eta}$ by (11). Then Lemma 3 yields $\mathcal{J}(\tilde{\eta}) = d_m \leqslant 0$. The strict inequality would contradict $\mathcal{J} \geqslant 0$, so that none of the intervals $(m, m+1]$ is strictly coupled with 0, in particular for $m = N$. Let now $m \in [1, N-1]$. If $\mathcal{J}(\tilde{\eta}) = 0$, then $\tilde{\eta}$ is optimal for $\mathcal{J}$, and hence must satisfy the Euler–Lagrange equation $(\mathcal{T})$ on $[0, N-1]$. Since $\tilde{\eta}_k = \eta_k$ on $[0, m]$, i.e., at least at two points, we must have $\tilde{\eta}_k = \eta_k$ for all $k \in J^*$. In particular, $\eta_k = \tilde{\eta}_k = \alpha$ on $[m+1, N+1]$, which is a contradiction. Therefore, for $m \in [1, N-1]$ the interval $(m, m+1]$ is not coupled with 0. Finally, Proposition 1 yields that $(0, 1]$ is not conjugate to 0.

"(iii) $\Rightarrow$ (iv)" Let X be the solution of $(\mathcal{T})$ satisfying (5). Conditions (18) and (19) follow from (iii) by Proposition 1, since no coupled intervals implies no conjugate intervals. If (20) does not hold, then there exist vectors $d \in \mathbb{R}^n$, $d \neq 0$, and $\beta \in \mathbb{R}^r$, such that

$$d^T X_N^T S_N X_{N+1} d - d^T X_N^T S_N Y \beta - \beta^T Y^T S_N^T X_N d + \beta^T Y^T T_{N+1} Y \beta < 0. \tag{28}$$

Define $\eta_k := X_k d$ for all $k \in J^*$ and $\alpha := Y\beta$. Then η solves $(\mathcal{T})$, satisfies (3), and $\eta_N \neq 0$. Moreover, the expression in (28) equals to d_N, so that $(N, N+1]$ is strictly coupled with 0, which contradicts (iii).

"(iv) $\Rightarrow$ (v)" Note first that (21) and (22) follow from (iv) by Proposition 1. From [18, Eq. (26)] we also have the identity

$$X_N^T S_N X_{N+1} = X_N^T \big(T_N - S_{N-1}^T D_{N-1} S_{N-1}\big) X_N. \tag{29}$$

Therefore, (20) implies (23).

"(v) $\Rightarrow$ (vi)" The inequalities $H_k > 0$ for all $k \in [0, N-1]$ follow from (v) and $H_k = D_k^{-1}$, $k \in [1, N-1]$, by Proposition 1. Thus, condition (27) is the same as (23). Also, (27) implies $H_N \geqslant 0$.

"(vi) $\Rightarrow$ (ii)" The inequalities $\mathcal{L}_k > 0$ for all $k \in [0, N-1]$ follow from (vi) by Proposition 1. Since $\mathcal{L}_{N-1}$ is invertible, it is known that $\mathcal{L}_{N+1} \geqslant 0$ iff

$$\mathcal{Z} := \begin{pmatrix} T_N & -S_N Y \\ -Y^T S_N^T & Y^T T_{N+1} Y \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ \vdots & \vdots \\ -S_{N-1} & 0 \end{pmatrix}^T \mathcal{L}_{N-1}^{-1} \begin{pmatrix} 0 & 0 \\ \vdots & \vdots \\ -S_{N-1} & 0 \end{pmatrix} \geqslant 0.$$

In the second term above we actually need to know only the right lower block of $\mathcal{L}_{N-1}^{-1}$, and a direct matrix computation shows that

$$\mathcal{Z} = \begin{pmatrix} T_N & -S_N Y \\ -Y^T S_N^T & Y^T T_{N+1} Y \end{pmatrix} - \begin{pmatrix} S_{N-1}^T H_{N-1}^{-1} S_{N-1} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} H_N & -S_N Y \\ -Y^T S_N^T & Y^T T_{N+1} Y \end{pmatrix}.$$

Thus, (27) gives $\mathcal{Z} \geqslant 0$. Therefore, $\mathcal{L}_{N+1} \geqslant 0$ follows and thus also $\mathcal{L}_N \geqslant 0$. The proof is now complete. □

In condition (iii) above we used the regularity of $\mathcal{J}$, the coupled intervals condition on $(1, N]$, and strictly coupled condition at $(N, N+1]$. As we shall see below, $\mathcal{J} \geqslant 0$ is equivalent to a simpler form of (iii) in terms of conjugate intervals.

Corollary 1. *Each condition in Theorem* 2 *is also equivalent to*

(iii)′ *There is no interval* $(m, m+1] \subseteq (0, N]$ *conjugate to* 0, *and* $(N, N+1]$ *is not strictly coupled with* 0.

Proof. "(iii) ⇒ (iii)′" By Remark 5, if the interval $(m, m+1]$ is not coupled with 0, it is also not conjugate to 0.

"(iii)′ ⇒ (iv)" If $(0, N]$ contains no interval conjugate to 0, then the solution X of $(\mathcal{T})$ given by the initial conditions (5) satisfies (18) and (19), by (iii) ⇒ (iv) in Proposition 1. Next, as in the proof of (iii) ⇒ (iv) in Theorem 2, $(N, N+1]$ not being strictly coupled with 0 implies (20). □

Remark 7. In general we could have $(0, 1]$ coupled with 0 when $\mathcal{J} \geqslant 0$. However, if we assume in addition to (17) a kind of "super strengthened Legendre condition" at $k = 0$,

$$\alpha_0^T T_0 \alpha_0 - \alpha_0^T S_0 \alpha > 0 \quad \text{for all nonzero } (\alpha_0, \alpha) \text{ with } M_0 \alpha_0 = 0,\ M\alpha = 0, \tag{30}$$

then $\mathcal{J} \geqslant 0$ implies that $(0, 1]$ is not coupled with 0. Observe that (30) is trivial when the left endpoint is fixed, and also under (17), when the right endpoint is fixed.

Proof. Indeed, if $(0,1]$ were coupled with 0, there exists η a solution of $(\mathcal{T})$ satisfying (3), $\eta_0 \neq 0$, and for some $\alpha \in \operatorname{Ker} M$, $\eta_k \not\equiv \alpha$ on $[1, N+1]$ and $d_0(\eta_0, \eta_1, \alpha) \leqslant 0$. Set $\tilde{\eta}$ such that $\tilde{\eta}_0 = \eta_0$, and $\tilde{\eta}_k = \alpha$ on $[1, N+1]$. It results that $\tilde{\eta}$ is admissible for $\mathcal{J}$ with $\mathcal{J}(\tilde{\eta}) = d_0(\eta_0, \eta_1, \alpha) \leqslant 0$. The nonnegativity of $\mathcal{J}$ yields that $\tilde{\eta}$ is optimal for $\mathcal{J}$ and thus satisfies, for some vector $\bar{\gamma}_0 \in \mathbb{R}^{r_0}$, the initial transversality condition

$$\alpha = S_0^{-1}\big(T_0\eta_0 + M_0^T\bar{\gamma}_0\big).$$

Hence, $\eta_0^T S_0\alpha = \eta_0^T T_0\eta_0$. From (30) we get that $\eta_0 = 0$, which is a contradiction. $\square$

Remark 8. The conditions of Theorem 2 only imply $Y^T T_{N+1} Y \geqslant 0$. However, if in addition to that $Y^T T_{N+1} Y$ is invertible, then conditions (20), (23), and (27) can be replaced by equivalent conditions, respectively,

$$\begin{aligned}
&X_N^T S_N X_{N+1} - X_N^T S_N Y\big(Y^T T_{N+1} Y\big)^{-1} Y^T S_N^T X_N \geqslant 0,\\
&T_N - S_{N-1}^T D_{N-1} S_{N-1} - S_N Y\big(Y^T T_{N+1} Y\big)^{-1} Y^T S_N^T \geqslant 0,\\
&H_{N+1} := H_N - S_N Y\big(Y^T T_{N+1} Y\big)^{-1} Y^T S_N^T \geqslant 0. \qquad (31)
\end{aligned}$$

In the following we show that if $(N, N+1]$ is not strictly coupled (resp. coupled) with 0, then Definition 2 can be simplified. To be more specific, similarly as in the countinous time setting [22, Lemma 3], the statement $\eta_k \not\equiv \alpha$ on $[m+1, N+1]$ can be eliminated.

Lemma 4. *Assume that* $(N, N+1]$ *is not strictly coupled* (*resp. not coupled*) *with* 0. *Then* $(m, m+1]$ *is strictly coupled* (*resp. coupled*) *with* 0 *iff*

$$\begin{aligned}
&\text{there exists a solution } \eta \text{ of } (\mathcal{T}) \text{ and (3) and } \alpha \in \operatorname{Ker} M \text{ such that}\\
&\eta_m \neq 0 \quad \text{and} \quad d_m(\eta_m, \eta_{m+1}, \alpha) < 0 \quad (\text{resp. } d_m \leqslant 0).
\end{aligned} \qquad (32)$$

Proof. The result follows trivially when $m = N$. Assume $m < N$. Part "$\Rightarrow$" of the result is immediate, so that we will prove "$\Leftarrow$" part. Suppose that $(N, N+1]$ is not strictly coupled (resp. not coupled) with 0 and that (32) holds for some η and α. We shall show that $\eta_k \not\equiv \alpha$ on $[m+1, N+1]$. If $\eta_k \equiv \alpha$ on $[m+1, N+1]$, we consider two cases: (a) $\alpha = 0$, and (b) $\alpha \neq 0$. For the case (a), since $\eta_k \equiv 0$ on $[m+1, N+1]$, $(\mathcal{T})$ yields that $\eta_k \equiv 0$ on J^* and hence a contradiction with $\eta_m \neq 0$ is obtained. For part (b), since $\eta_k \equiv \alpha$ on $[m+1, N+1]$, then $d_m = d_m(\eta_m, \alpha, \alpha) = -\eta_m^T S_m\alpha + \alpha^T \widetilde{T}_{m+1}\alpha$. Using the fact that η satisfies $(\mathcal{T})$ on $[m, N-1]$ we obtain $d_m = \alpha^T (T_{N+1} - S_N^T)\alpha = d_N$. Since $d_m < 0$ (resp. $d_m \leqslant 0$) we get $(N, N+1]$ is strictly coupled (resp. coupled) with 0, contradicting the assumption. Therefore, $\eta_k \not\equiv \alpha$ on $[m+1, N+1]$, and the result is proved. $\square$

The following provides a connection between the regularity (resp. strong regularity) and $(0,1]$ not strictly coupled (resp. not coupled) with 0.

Corollary 2. *Assume the Legendre condition $Y^T \widetilde{T}_1 Y \geqslant 0$ holds. If $(N, N+1]$ is not strictly coupled (resp. not coupled) with 0, then $(0,1]$ is not strictly coupled (resp. not coupled) with 0 iff $\mathcal{J}$ is regular (resp. strongly regular).*

Proof. "$\Rightarrow$" Let $\eta_0, \alpha \in \mathbb{R}^n$ be such that $M_0\eta_0 = 0$, $M\alpha = 0$. If $\eta_0 = 0$, then $\mathcal{J}_0(0,\alpha) = \alpha^T \widetilde{T}_1 \alpha \geqslant 0$. If $\eta_0 \neq 0$, then set $\eta_1 := S_0^{-1}(T_0\eta_0 + M_0^T \gamma_0)$ for some $\gamma_0 \in \mathbb{R}^{r_0}$. Then construct $\{\eta_k\}_{k=0}^{N+1}$ as a solution of $(\mathcal{T})$ by

$$\eta_{k+2} = S_{k+1}^{-1}\big(T_{k+1}\eta_{k+1} - S_k^T \eta_k\big), \quad k \in [0, N-1].$$

By Lemma 4 and the hypothesis that $(0,1]$ is not strictly coupled (resp. not coupled) with 0 we have $d_0(\eta_0, \eta_1, \alpha) \geqslant 0$ (resp. $d_0 > 0$). Using the definition of η_1 and d_0 we get

$$d_0(\eta_0, \eta_1, \alpha) = \eta_0^T T_0 \eta_0 - \eta_0^T S_0 \alpha - \alpha^T S_0^T \eta_0 + \alpha^T \widetilde{T}_1 \alpha = \mathcal{J}_0(\eta_0, \alpha).$$

Hence, $\mathcal{J}$ is regular (resp. strongly regular).

"$\Leftarrow$" Assume $\mathcal{J}$ is regular (resp. strongly regular). Let η, α be such that η solves $(\mathcal{T})$ and (3), $\eta_0 \neq 0$. Then, by Lemma 3 with $m = 0$, $d_0(\eta_0, \eta_1, \alpha) = \mathcal{J}_0(\eta_0, \alpha) \geqslant 0$ (resp. $d_0 > 0$). Therefore $(0,1]$ is not strictly coupled (resp. not coupled) with 0. $\square$

Remark 9. From Corollary 2 it results that in Theorem 2(iii) the condition that $\mathcal{J}$ is regular can be replaced by the condition that $(0,1]$ is not strictly coupled with 0.

In this result we show that the nonnegativity of $\mathcal{J}$ is equivalent to the positivity of $\mathcal{J}_m(\eta, \alpha)$ over all (η, α) admissible with α not equal to a unique value depending only on η.

Theorem 3 (Characterization of $\mathcal{J} \geqslant 0$). *Assume* (A1) *and suppose that* (17) *holds. Then $\mathcal{J} \geqslant 0$ is also equivalent to*

(vii) *For all $m \in J$, $\mathcal{J}_m(\eta, \alpha) > 0$ for all admissible η and α, not both zero, with $\alpha \neq \alpha_m$ (drop if $M = I$), where*

$$\alpha_m := S_m^{-1}\big(T_m\eta_m - S_{m-1}^T \eta_{m-1}\big). \tag{33}$$

Proof. We will show that (vii) is equivalent to $\mathcal{J}_m \geqslant 0$ for all $m \in J$, which in turn is equivalent to $\mathcal{J} \geqslant 0$ by Lemma 2. Suppose first that $\mathcal{J}_m \geqslant 0$ for all $m \in J$ and fix $m \in J$. If $M = I$, Lemma 2 and Proposition 1 yield (vii). If $M \neq I$ and $\mathcal{J}_m(\eta, \alpha) = 0$ for some admissible $\{\eta_k\}_{k=0}^{m}$ and $\alpha \neq \alpha_m$, then $\eta = \{\eta_k\}_{k=0}^{m+1}$

where $\eta_{m+1} = \alpha$ is optimal for the accessory problem with $\mathcal{J}_m$. Hence, there exist vectors $\tilde{\gamma}_0 \in \mathbb{R}^{r_0}$ and $\tilde{\gamma} \in \mathbb{R}^r$ such that η satisfies the Euler–Lagrange equation $(\mathcal{T})$ on $[0, m-1]$, the initial boundary and tranversality conditions

$$M_0 \tilde{\eta}_0 = 0, \quad \tilde{\eta}_1 = S_0^{-1}\big(T_0 \tilde{\eta}_0 + M_0^T \tilde{\gamma}_0\big),$$

at $(0, 1]$, and also the transversality condition

$$\eta_m = S_m^{T-1}\big(\widetilde{T}_{m+1}\alpha + M^T \tilde{\gamma}\big), \tag{34}$$

at $(m, m+1]$, where $\widetilde{T}_{m+1}$ is defined by (9). In particular, the Euler–Lagrange equation $(\mathcal{T})$ at $k = m-1$ gives

$$-S_m \alpha + T_m \eta_m - S_{m-1}^T \eta_{m-1} = 0,$$

from which we obtain the value $\alpha = \alpha_m$, which is a contradiction. Thus we must have $\mathcal{J}_m(\eta, \alpha) > 0$.

Conversely, assume (vii) and fix $m \in J$. Let η and α be admissible for $\mathcal{J}_m$. If $M = I$, then $\alpha = 0$ and $\mathcal{J}_m(\eta, 0) \geqslant 0$ follows from Proposition 1. Thus, we suppose in the remaining part that $M \neq I$. If $\alpha \neq \alpha_m$, then $\mathcal{J}_m(\eta, \alpha) \geqslant 0$ by the assumption. If $\alpha = \alpha_m$, we write $\alpha = Y\beta$ for some nonzero $\beta \in \mathbb{R}^r$, in fact $\beta = Y^T \alpha$. Pick any sequence $\{\beta_j\}_{j=0}^{\infty}$ of vectors in $\mathbb{R}^r$ with the property that $\beta_j \neq \beta$ for all j and $\beta = \lim_{j\to\infty} \beta_j$. Then it follows that $Y\beta_j \neq \alpha$ and thus $\mathcal{J}_m(\eta, Y\beta_j) > 0$, by the assumption. Taking the limit as $j \to \infty$ we obtain

$$\mathcal{J}_m(\eta, \alpha_m) = \mathcal{J}_m(\eta, Y\beta) = \lim_{j\to\infty} \mathcal{J}_m(\eta, Y\beta_j) \geqslant 0.$$

Therefore, $\mathcal{J}_m \geqslant 0$ and the proof is complete. □

Remark 10. Condition $\mathcal{J} \geqslant 0$ implies that $Y^T \widetilde{T}_{m+1} Y \geqslant 0$ for all $m \in J$, by Theorem 1. If moreover $Y^T \widetilde{T}_{m+1} Y$ is invertible, one can obtain the value α_m also from the transversality condition (34) as

$$\tilde{\alpha}_m := Y\big(Y^T \widetilde{T}_{m+1} Y\big)^{-1} Y^T S_m^T \eta_m.$$

Obviously, in this case α_m defined in (33) must be the same as $\tilde{\alpha}_m$ above.

Remark 11. The above result suggests defining a coupled interval with 0 via $\mathcal{J}_m$. Indeed, the following is a definition equivalent to Definition 2: *Let $m \in J$. An interval $(m, m+1]$ is coupled with 0 if there exists a solution $\{\eta_k\}_{k=0}^{m}$ to the Euler–Lagrange equation $(\mathcal{T})$, $k \in [0, m-2]$, satisfying the initial boundary and transversality conditions* (3), *$\eta_m \neq 0$, and for some $\alpha \in \operatorname{Ker} M$ we have $\mathcal{J}_m(\eta, \alpha) \leqslant 0$.*

The following result shows that a natural and slight strengthening of each of the conditions characterizing $\mathcal{J} \geqslant 0$ actually gives a characterization of $\mathcal{J} > 0$. In the statement below X is the solution of $(\mathcal{T})$ given by the initial conditions (5). Recall that $\mathcal{J} > 0$ implies the strengthened Legendre condition (14), and in particular (17) and $Y^T T_{N+1} Y > 0$ hold.

Theorem 4 (Characterization of $\mathcal{J} > 0$). *Suppose* (A1) *holds. Then the following are equivalent.*

(i) $\mathcal{J} > 0$, *i.e.*, $\mathcal{J}(\eta) > 0$ *for all admissible* η, $\eta \not\equiv 0$.
(ii) $\mathcal{J} \geqslant 0$, (17) *holds*, $Y^T T_{N+1} Y$ *is invertible if* $M \neq I$, *and any of the following conditions is satisfied.*
  (a) X_{N+1} *is invertible.*
  (b) H_N *is invertible.*
  (c) $(N, N+1]$ *is not coupled with* 0.
(iii) $\mathcal{L}_k > 0$ *for all* $k \in J^*$.
(iv) *There is no interval* $(m, m+1] \subseteq (0, N+1]$ *coupled with* 0.
(v) X_k *is invertible for all* $k \in [1, N+1]$, (18) *holds*, $X_k^T S_k X_{k+1} > 0$ *for all* $k \in [1, N]$, *and*

$$X_N^T S_N X_{N+1} - X_N^T S_N Y \big(Y^T T_{N+1} Y\big)^{-1} Y^T S_N^T X_N > 0. \tag{35}$$

(vi) X_k *is invertible for all* $k \in [1, N+1]$, (21) *holds*, $D_k > 0$ *for all* $k \in [1, N]$, *and*

$$T_{N+1} - S_N^T D_N S_N > 0 \quad \textit{on } \operatorname{Ker} M. \tag{36}$$

(vii) $H_k > 0$ *for all* $k \in J^*$, *where* H_{N+1} *is defined in* (31).
(viii) $\mathcal{J}_m > 0$ *for all* $m \in J$.

Proof. The proof follows the proof of Theorem 2, where each condition at $(N, N+1]$ is strengthened. Note first that (i) and (viii) are equivalent because of $\mathcal{J} = \mathcal{J}_N$. We will show briefly the implications

$$\text{(iii)} \Rightarrow \text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(iv)} \Rightarrow \text{(v)} \Rightarrow \text{(vi)} \Rightarrow \text{(vii)} \Rightarrow \text{(iii)}.$$

"(iii) $\Rightarrow$ (i)" This is a direct consequence of Lemma 1.

"(i) $\Rightarrow$ (ii)" $\mathcal{J} > 0$ implies $\mathcal{J} \geqslant 0$ and, by Lemma 1, conditions (17) and $Y^T T_{N+1} Y > 0$ hold true. Moreover, $\mathcal{J} > 0$ implies that X_{N+1} is invertible. For if $X_{N+1} c = 0$, then $\eta_k := X_k c$ is a solution of $(\mathcal{T})$, which is admissible for $\mathcal{J}$ and which satisfies $\mathcal{J}(\eta) = 0$. Now if $\eta \not\equiv 0$, we have a contradiction with $\mathcal{J} > 0$. Thus, $\eta_k \equiv 0$ on J^*. In particular, from (5) we have $X_0 c = (I - \mathcal{M}_0) c = 0$. Moreover, (5) also yields $S_0 X_1 c = T_0 X_0 c + \mathcal{M}_0 c$, i.e., $\mathcal{M}_0 c = 0$. Hence it follows that $c = 0$ and X_{N+1} is invertible, so that part (a) is proven. To show (b), recall that $\mathcal{J} \geqslant 0$ implies $X_N^T S_N X_{N+1} \geqslant 0$, by Theorem 2. Since all the matrices in the last product are invertible, we must have $X_N^T S_N X_{N+1} > 0$, which is equivalent to $T_N - S_{N-1}^T D_{N-1} S_{N-1} > 0$, which in turn is equivalent to $H_N > 0$. For (c), if $Y^T T_{N+1} Y$ is invertible, then $\mathcal{J} \geqslant 0$ implies $Y^T T_{N+1} Y > 0$, by Theorem 1. Then, from Remark 8 we obtain $H_{N+1} > 0$, which is equivalent to (35). Suppose now that $(N, N+1]$ is coupled with 0 with the corresponding solution η and α. Then

$\eta_k = X_k c$ for some $c \in \mathbb{R}^n$. Moreover, $d_N = 0$ just because $(N, N+1]$ cannot be strictly coupled with 0, by Theorem 2. But

$$0 = d_N = c^T X_N^T S_N X_{N+1} c - c^T X_N^T S_N \alpha - \alpha^T S_N^T X_N c + \alpha^T T_{N+1} \alpha,$$

which contradicts (35). Thus, $(N, N+1]$ is not coupled with 0. Finally, to show (c) $\Rightarrow$ (a), if $(N, N+1]$ is not coupled with 0, it is not conjugate to 0 so that X_{N+1} must be invertible.

"(ii)(c) $\Rightarrow$ (iv)" This is a consequence of Theorem 2.

"(iv) $\Rightarrow$ (v)" Theorem 2 yields that X_k is invertible on $[1, N]$, and we have seen above that $(N, N+1]$ not coupled with 0 implies X_{N+1} invertible. Also, since $(0, 1]$ is not coupled with 0, it is not conjugate to 0, and thus (18) follows. Condition (19) follows from Theorem 2, and together with the invertibility of X_{N+1} it implies $X_N^T S_N X_{N+1} > 0$. To prove (35), note that $(N, N+1]$ not coupled with zero means that $d_N > 0$ for any solution η of $(\mathcal{T})$ with corresponding α, $(\eta, \alpha) \neq 0$. Thus for $\eta \equiv 0$ and $\alpha \neq 0$ we have $d_N = \alpha^T T_{N+1} \alpha > 0$. Therefore, $Y^T T_{N+1} Y > 0$. Condition (35) then follows from (20).

"(v) $\Rightarrow$ (vi)" From Theorem 2 we have X_k invertible on $[1, N+1]$, (21) holds, and $D_k > 0$ for all $k \in [1, N]$. Moreover,

$$\begin{aligned} &D_N^{-1} - S_N Y \left(Y^T T_{N+1} Y\right)^{-1} Y^T S_N^T \\ &\quad = T_N - S_{N-1}^T D_{N-1} S_{N-1} - S_N Y \left(Y^T T_{N+1} Y\right)^{-1} Y^T S_N^T > 0 \end{aligned}$$

implies $D_N > 0$. The above inequality can be replaced by an equivalent condition

$$\begin{pmatrix} D_N^{-1} & -S_N Y \\ -Y^T S_N^T & Y^T T_{N+1} Y \end{pmatrix} > 0,$$

which is also equivalent to (36).

"(vi) $\Rightarrow$ (vii)" We have $H_k = D_k^{-1} > 0$ for all $k \in J$. Furthermore, (36) implies $H_{N+1} > 0$.

"(vii) $\Rightarrow$ (iii)" By Theorem 2, conditions $H_k > 0$ for all $k \in J$ imply $\mathcal{L}_k > 0$ for all $k \in J$. But since in this case the matrix $\mathcal{Z}$ in the proof of Theorem 2 satisfies $\mathcal{Z} > 0$, we have $\mathcal{L}_{N+1} > 0$. The proof is complete. $\square$

Remark 12. From Corollary 2 it results that condition (iv) in Theorem 4 can be replaced by the condition that $\mathcal{J}$ is strongly regular and there is no interval $(m, m+1] \subseteq (1, N+1]$ coupled with 0.

Remark 13. Theorem 4 says that, under the initial strengthened Legendre condition (17), the gap between $\mathcal{J} \geqslant 0$ and $\mathcal{J} > 0$ is as close, or as far, as X_{N+1} being invertible. Thus, the equivalence of (i) and (iii) in Theorem 2 can be interpreted as a discrete analogue of the continuous time statement (with the notation from Section 1)

$$\mathcal{G}_2 \geqslant 0 \;\Leftrightarrow\; \mathcal{G}_2 \text{ is regular and there is no point } c \in [a, b) \text{ coupled with } a,$$

which holds under the strengthened Legendre condition $R(t) > 0$, see, e.g., [25].

4. Riccati equation for $\mathcal{I} \geqslant 0$ and $\mathcal{I} > 0$

Next we wish to include a result on the Riccati matrix difference equation $(\mathcal{R})$. There is an extensive literature on this subject arising in the control theory and, in particular, in the calculus of variations, see, e.g., [3] and the references therein. Some of those papers [2,6,7,10] focused on the question of a characterization of the *positivity* of quadratic functionals in terms of $(\mathcal{R})$. However, until recently, no work was devoted to the question of the *nonnegativity* of $\mathcal{I}$ in terms of $(\mathcal{R})$, where $\mathcal{I}$ is the discrete quadratic functional introduced in Section 1. The first work in this direction, as far as we know, is [18], where the result for the fixed right endpoint case was established. In this section we extend this characterization to the variable endpoints given by (1).

The Riccati equation can be derived only when the quadratic functional $\mathcal{J}$ corresponds to the quadratic functional $\mathcal{I}$ with R_k invertible. Expanding the forward differences, we obtain $\mathcal{J}(\eta) = \mathcal{I}(\eta)$. Equation $(\mathcal{T})$ is then the *Jacobi difference equation*

$$\Delta\big(R_k \Delta\eta_k + Q_k^T \eta_{k+1}\big) = Q_k \Delta\eta_k + P_k \eta_{k+1}, \quad k \in [0, N-1], \tag{37}$$

and the matrices T_k and S_k are defined by

$$\begin{aligned}
&T_0 := \Gamma_0 + R_0,\\
&T_{k+1} := R_k + R_{k+1} + Q_k + Q_k^T + P_k, \quad k \in [0, N-1],\\
&T_{N+1} := \Gamma + R_N + Q_N + Q_N^T + P_N,\\
&S_k := R_k + Q_k^T, \quad k \in J.
\end{aligned} \tag{38}$$

In this section we will use the assumption

$$S_k,\ R_k \text{ invertible for all } k \in J, \text{ and } M_0, M \text{ have full rank.} \tag{A2}$$

The quantity d_m in the definition of coupled interval with 0 can be written in this case as

$$\begin{aligned}
d_m = {} & \eta_m^T S_m \eta_{m+1} - \eta_m^T S_m \alpha - \alpha^T S_m^T \eta_m\\
& + \alpha^T \left\{ \Gamma + R_m + Q_m + Q_m^T + \sum_{k=m}^{N} P_k \right\} \alpha.
\end{aligned}$$

Remark 14. Denote $\widehat{\Gamma}_0 := (I - \mathcal{M}_0)\Gamma_0(I - \mathcal{M}_0)$ and $\widehat{\Gamma} := (I - \mathcal{M})\Gamma(I - \mathcal{M})$. Then it follows that $\eta_0^T \Gamma \eta_0 = \eta_0^T \widehat{\Gamma}_0 \eta_0$ for $\eta_0 \in \operatorname{Ker} \mathcal{M}_0$, and similarly for $\widehat{\Gamma}$ and $\eta_{N+1} \in \operatorname{Ker} \mathcal{M}$. Thus, we may replace Γ_0 and Γ by $\widehat{\Gamma}_0$ and $\widehat{\Gamma}$, respectively.

If (A2) holds, Jacobi equation (37) can be rewritten as a linear Hamiltonian system $(\mathcal{H})$ with $U_k = R_k \Delta X_k + Q_k^T X_{k+1}$, where

$$A_k := -R_k^{-1} Q_k^T, \qquad B_k := R_k^{-1}, \qquad C_k := P_k - Q_k R_k^{-1} Q_k^T. \tag{39}$$

In this section we use the U_k above when X_k is given. Then X solves (37) iff (X, U) solves $(\mathcal{H})$, and one also has $\mathcal{I}(\eta) = \mathcal{K}(\eta, q)$ with

$$q_k := R_k \Delta\eta_k + Q_k^T \eta_{k+1}. \tag{40}$$

By using (39), a direct computation also shows that the Riccati matrix equation $(\mathcal{R})$ takes in this case the form

$$\begin{aligned} \underline{R}[W]_k &:= \Delta W_k - P_k + (W_{k+1} - P_k - Q_k)\big(R_k + Q_k^T\big)^{-1}\big(W_k - Q_k^T\big) \\ &= 0. \end{aligned} \tag{$\underline{\mathcal{R}}$}$$

Although R_k^{-1} does not appear in $(\underline{\mathcal{R}})$ directly, we needed R_k invertible to derive it.

Theorem 5. *Let T_k, S_k be defined by* (38) *and assume* (A2). *Then conditions* (i)–(vi) *of Theorem* 2 *and* (vii) *of Theorem* 3 *are also equivalent to*

(viii) *There exists a symmetric solution W_k on $[0, N]$ to the Riccati matrix difference equation $(\underline{\mathcal{R}})$, $k \in [1, N-1]$, with $\underline{R}[W]_0 Y_0 = 0$, such that*

$$W_0 = \widehat{\Gamma}_0, \tag{41}$$

$$Y_0^T (R_0 + W_0) Y_0 > 0, \tag{42}$$

$$R_k + W_k > 0 \quad \text{for all } k \in [1, N-1], \tag{43}$$

$$\begin{pmatrix} R_N + W_N & -(R_N + Q_N^T)Y \\ -Y^T(R_N + Q_N) & Y^T T_{N+1} Y \end{pmatrix} \geqslant 0 \tag{44}$$

and the matrix $\mathcal{D}_0$ defined by

$$\mathcal{D}_0 := (R_0 + Q_0)^{-1}\big(R_0 + Q_0 + Q_0^T + P_0 - W_1\big)\big(R_0 + Q_0^T\big)^{-1}$$

satisfies

$$\mathcal{D}_0 \mathcal{M}_0 = 0. \tag{45}$$

Proof. "Theorem 2(v) $\Rightarrow$ (viii)". All the conditions of (viii) follow from (v) of Theorem 2 by [18, Theorem 3], except of (44). However, since

$$X_N^T\big(T_N - S_{N-1}^T D_{N-1} S_{N-1}\big)X_N = X_N^T S_N X_{N+1} = X_N^T (R_N + W_N) X_N,$$

inequality (44) follows immediately from (23).

"(viii) $\Rightarrow$ Theorem 2(i)". Let $\{\eta_k\}_{k=0}^{N+1}$ be admissible for $\mathcal{I}$, $\eta_{N+1} = Y\beta$, and define q_k by (40), so that (η, q) is admissible for $\mathcal{K}$ and $\mathcal{K}(\eta, q) = \mathcal{I}(\eta)$. Note that for the matrices $\mathcal{D}_k$ defined by

$$\mathcal{D}_k = B_k - B_k \tilde{A}_k^T (W_{k+1} - C_k) \tilde{A}_k B_k, \quad k \in [0, N-1],$$

we have $\mathcal{D}_0 \geqslant 0$ and $\mathcal{D}_k > 0$, $k \in [1, N-1]$, by the proof of [18, Theorem 3]. Furthermore, the equation $\Delta\eta_N = A_N \eta_{N+1} + B_N q_N$ implies

$$q_N = \big(R_N + Q_N^T\big)\eta_{N+1} - R_N \eta_N. \tag{46}$$

Since $R[W]_0 Y_0 = 0$ and $R[W]_k = 0$, $k \in [1, N-1]$, we apply a Picone-type identity, more precisely [6, Lemma 2(i)], on the interval $[0, N-1]$. Set $z_k := q_k - W_k\eta_k$, $k \in J$, then

$$\begin{aligned}
\mathcal{I}(\eta) = \mathcal{K}(\eta, q) &= \eta_0^T \Gamma_0 \eta_0 + \eta_{N+1}^T \Gamma \eta_{N+1} \\
&\quad + \sum_{k=0}^{N} \left\{ \eta_{k+1}^T C_k \eta_{k+1} + q_k^T B_k q_k \right\} \\
&= \eta_0^T (\Gamma_0 - W_0)\eta_0 + \eta_{N+1}^T \Gamma \eta_{N+1} + \eta_N^T W_N \eta_N + \sum_{k=0}^{N-1} z_k^T \mathcal{D}_k z_k \\
&\quad + \left\{ \eta_{N+1}^T C_N \eta_{N+1} + q_N^T B_N q_N \right\} \\
&\overset{(41),(46)}{\geqslant} \eta_N^T W_N \eta_N + \eta_{N+1}^T \left(\Gamma + P_N - Q_N R_N^{-1} Q_N^T \right) \eta_{N+1} \\
&\qquad + \left\{ \eta_{N+1}^T (R_N + Q_N) - \eta_N^T R_N \right\} \\
&\qquad \times R_N^{-1} \left\{ \left(R_N + Q_N^T \right) \eta_{N+1} - R_N \eta_N \right\} \\
&= \eta_N^T (R_N + W_N) \eta_N - \eta_N^T \left(R_N + Q_N^T \right) \eta_{N+1} \\
&\quad - \eta_{N+1}^T (R_N + Q_N) \eta_N + \eta_{N+1}^T T_{N+1} \eta_{N+1} \\
&= \begin{pmatrix} \eta_N \\ \beta \end{pmatrix}^T \begin{pmatrix} R_N + W_N & -(R_N + Q_N^T)Y \\ -Y^T (R_N + Q_N) & Y^T T_{N+1} Y \end{pmatrix} \begin{pmatrix} \eta_N \\ \beta \end{pmatrix} \overset{(44)}{\geqslant} 0.
\end{aligned}$$

Therefore, $\mathcal{I}(\eta) \geqslant 0$ and the proof is now complete. □

Remark 15. Similarly to Remark 8, if $Y^T T_{N+1} Y$ is invertible, condition (44) can be replaced by an equivalent condition

$$R_N + W_N - \left(R_N + Q_N^T \right) Y \left(Y^T T_{N+1} Y \right)^{-1} Y^T (R_N + Q_N) \geqslant 0.$$

Theorem 6. *Let* T_k, S_k *be defined by* (38) *and assume* (A2). *Conditions* (i)–(viii) *of Theorem* 4 *are also equivalent to*

(ix) *There exists a symmetric solution* W_k *on* J^* *of the Riccati matrix equation* ($\underline{\mathcal{R}}$), $k \in [1, N]$, *with* $\underline{R}[W]_0 Y_0 = 0$, $\mathcal{D}_0 \mathcal{M}_0 = 0$, *which satisfies* (41), (42), $R_k + W_k > 0$ *for all* $k \in [1, N]$, *and*

$$T_{N+1} - (R_N + Q_N)(R_N + W_N)^{-1} \left(R_N + Q_N^T \right) > 0 \quad \textit{on } \operatorname{Ker} M. \tag{47}$$

Proof. In addition to the proof of (viii) ⇒ (i) in Theorem 5, we need to show that $\mathcal{I}(\eta) = 0$ implies $\eta_k \equiv 0$ for all $k \in J^*$. However, if $\mathcal{I}(\eta) = 0$ then $\mathcal{D}_k z_k = 0$, $k \in J$, and thus for $\eta_{N+1} = Y\beta$,

$$\mathcal{I}(\eta) = \begin{pmatrix} \eta_N \\ \beta \end{pmatrix}^T \begin{pmatrix} R_N + W_N & -(R_N + Q_N^T)Y \\ -Y^T (R_N + Q_N) & Y^T T_{N+1} Y \end{pmatrix} \begin{pmatrix} \eta_N \\ \beta \end{pmatrix} = 0.$$

If $\eta_{N+1} \neq 0$, i.e., if $\beta \neq 0$, the above equality contradicts (47). Hence we must have $\eta_{N+1} = 0$. It follows from [6, Lemma 2(i)] that η satisfies $Z_k \eta_{k+1} = \eta_k$, $k \in J$, where Z_k is a certain $n \times n$ matrix. Hence, $\eta_N = \cdots = \eta_0 = 0$. □

5. Optimality conditions

In this section we present necessary and sufficient optimality conditions for the variational problem (P). A sequence $x = \{x_k\}_{k=0}^{N+1}$ satisfying the boundary conditions in (P) is called *feasible* for (P). A feasible sequence $\hat{x}$ is called a *weak local minimum* for (P) if for some $\varepsilon > 0$, $\hat{x}$ minimizes $\mathcal{F}(x)$ over all feasible sequences x satisfying $|x_k - \hat{x}_k| < \varepsilon$, $k \in J^*$, where $|\cdot|$ is any norm in $\mathbb{R}^n$.

If $x = \{x_k\}_{k=0}^{N+1}$ is a weak local minimum for (P), then it is known, see, e.g., [1, 2,4] for the fixed endpoints and in [16,17] for the varying endpoints, that x must satisfy the Euler–Lagrange difference equation

$$\Delta g_u(k, x_{k+1}, \Delta x_k) = g_x(k, x_{k+1}, \Delta x_k), \quad k \in [0, N-1], \tag{48}$$

and, for some $\gamma_0 \in \mathbb{R}^{r_0}$, $\gamma \in \mathbb{R}^r$, the transversality conditions

$$g_u(0, x_1, \Delta x_0) = \nabla K_0(x_0) + \gamma_0^T M_0, \tag{49}$$

$$g_u(N, x_{N+1}, \Delta x_N) + g_x(N, x_{N+1}, \Delta x_N) + \nabla K(x_{N+1}) + \gamma^T M = 0, \tag{50}$$

where g_x, g_u are the gradients of g with respect to the second and the last variable, respectively. The second variation $\mathcal{F}_2$ is the quadratic functional $\mathcal{F}_2 = \frac{1}{2}\mathcal{I} \geqslant 0$, where

$$P_k := g_{xx}, \qquad Q_k := g_{xu}, \qquad R_k := g_{uu},$$

evaluated at $(k, x_{k+1}, \Delta x_k)$, and

$$M_0 := \nabla\varphi_0(x_0), \qquad \Gamma_0 := \nabla^2 K_0(x_0) + \gamma_0^T \nabla^2\varphi_0(x_0),$$
$$M := \nabla\varphi(x_{N+1}), \qquad \Gamma := \nabla^2 K(x_{N+1}) + \gamma^T \nabla^2\varphi(x_{N+1}).$$

The minimization problem for $\mathcal{F}_2$ over (1), known also as the *accessory problem*, yields the Jacobi difference equation (37) and the transversality conditions

$$\eta_1 = S_0^{-1}\big(T_0\eta_0 + M_0^T\gamma_0\big), \qquad \eta_N = S_N^{T-1}\big(T_{N+1}\eta_{N+1} + M^T\gamma\big).$$

Applying the results from Sections 3, 4 to $\mathcal{F}_2$ we obtain the following necessary and sufficient conditions for optimality in (P).

Theorem 7 (Necessary optimality conditions). *Let T_k, S_k be defined by* (38). *Assume* (A1) *and that the strengthened Legendre condition* (17) *holds. If $x = \{x_k\}_{k=0}^{N+1}$ is a weak local minimum for* (P)*, then all the conditions* (i)–(vi) *of Theorem* 2 *and the condition* (vii) *of Theorem* 3 *are satisfied. In addition, if* (A2) *holds, then the condition* (viii) *of Theorem* 5 *is also satisfied.*

Proof. Apply Theorems 2, 3, 5. □

Theorem 8 (Sufficient optimality conditions). *Let T_k, S_k be defined by* (38). *Suppose that $x = \{x_k\}_{k=0}^{N+1}$ is a feasible sequence for* (P) *satisfying the Euler–Lagrange equation* (48) *and, for some $\gamma_0 \in \mathbb{R}^{r_0}$, $\gamma \in \mathbb{R}^r$, the transversality conditions* (49)–(50). *If any of the conditions* (i)–(viii) *of Theorem* 4 *holds, or under* (A2) *the condition* (ix) *of Theorem* 6 *holds, then x is a strict weak local minimum for* (P).

Proof. Apply the result of [19, p. 307], or [17, Theorems 1, 3], with Theorems 4, 6. □

Remark 16. Together with our previous work [18], we have completed the set of the second order necessary and sufficient optimality conditions for the problem of the discrete calculus of variations, when either one endpoint is fixed of both are varying separately. One of the novelties of these results lies in the fact that both necessary and sufficient optimality criteria are phrased in terms of a coupled interval notion as well as in terms of the existence of a solution to the explicit Riccati equation ($\mathcal{R}$) with appropriate boundary conditions. The extension of these results to the case when both endpoints vary *jointly* is an interesting open problem. Note that this more general setting seems to require an approach different than the one employed in this paper. This will be a topic of our future work. It is worth mentioning that for the general setting, a characterization of the positivity of $\mathcal{J}$ can be deduced in [7] and [10] in terms of the existence of a solution W to the Riccati equation ($\mathcal{R}$) and a certain endpoints inequality (see Theorem 1(ii) of [10]). Unlike the results of this paper, the endpoints inequality used therein does not involve the endpoints of the solution W, but it is rather required to hold at *all* endpoints of the solutions (η, q) of the Hamiltonian system ($\mathcal{H}$) and the joint boundary conditions for η. Checking this condition would then require finding all such endpoints. However, for the case of separated endpoints, this paper provides an easier condition to check in terms of the boundary conditions of W (see Theorem 6).

Remark 17. From the notions of conjugate and coupled intervals for the discrete quadratic functional $\mathcal{I}$ we can derive the corresponding notions for the continuous-time quadratic problem. This could be done by introducing the *variable stepsize* discrete quadratic functional and Jacobi difference equation to which the results of this paper would be applied. Subsequently, by taking the limit as the stepsize decreases to 0, we obtain the continuous-time conjugate and coupled point notions. The details of this computation and other related results on the "convergence" of discrete criteria to continuous ones will be presented in a subsequent paper.

Acknowledgment

The authors wish to thank an anonymous referee for his/her constructive comments. In particular, we appreciate his/her question discussed in Remark 17.

References

[1] R.P. Agarwal, C.D. Ahlbrandt, M. Bohner, A. Peterson, Discrete linear Hamiltonian systems: A survey, in: R.P. Agarwal, M. Bohner (Eds.), Discrete and Continuous Hamiltonian Systems, Dynam. Systems Appl. 8 (3–4) (1999) 307–333.

[2] C.D. Ahlbrandt, Discrete variational inequalities, in: W. Walter (Ed.), General Inequalities 6, Proceedings of the 6th International Conference on General Inequalities, Oberwolfach, 1990, in: International Series of Numerical Mathematics, Vol. 123, Birkhäuser, Basel, 1992, pp. 93–107.

[3] C.D. Ahlbrandt, M. Heifetz, Discrete Riccati equations of filtering and control, in: S. Elaydi, J. Graef, G. Ladas, A. Peterson (Eds.), Proceedings of the First International Conference on Difference Equations, San Antonio, TX, 1994, Gordon and Breach, Newark, NJ, 1996, pp. 1–16.

[4] C.D. Ahlbrandt, J.W. Hooker, A variational view of non-oscillation theory for linear difference equations, in: J.L. Henderson (Ed.), Differential and Integral Equations, Proceedings of the Twelfth and Thirteenth Midwest Conferences, Iowa City, IO, 1983, Argonne, IL, 1984, Institute of Applied Mathematics, University of Missouri-Rolla, Rolla, MO, 1985, pp. 1–21.

[5] C.D. Ahlbrandt, A.C. Peterson, Discrete Hamiltonian Systems: Difference Equations, Continued Fractions, and Riccati Equations, Kluwer Academic Publishers, Boston, 1996.

[6] M. Bohner, Linear Hamiltonian difference systems: disconjugacy and Jacobi-type conditions, J. Math. Anal. Appl. 199 (1996) 804–826.

[7] M. Bohner, Riccati matrix difference equations and linear Hamiltonian difference systems, Dynam. Contin. Discrete Impuls. Systems 2 (2) (1996) 147–159.

[8] M. Bohner, O. Došlý, Disconjugacy and transformations for symplectic systems, Rocky Mountain J. Math. 27 (3) (1997) 707–743.

[9] M. Bohner, O. Došlý, Positivity of block tridiagonal matrices, SIAM J. Matrix Anal. 20 (1) (1998) 182–195.

[10] M. Bohner, O. Došlý, W. Kratz, Discrete Reid roundabout theorems, in: R.P. Agarwal, M. Bohner (Eds.), Discrete and Continuous Hamiltonian Systems, Dynam. Systems Appl. 8 (3–4) (1999) 345–352.

[11] Z. Došlá, O. Došlý, Quadratic functionals with general boundary conditions, Appl. Math. Optim. 36 (1997) 243–262.

[12] Z. Došlá, P. Zezza, Quadratic functionals with a variable singular end point, Comment. Math. Univ. Carolin. 33 (3) (1992) 411–425.

[13] L. Erbe, P. Yan, Disconjugacy for linear Hamiltonian difference systems, J. Math. Anal. Appl. 167 (1992) 355–367.

[14] L. Erbe, P. Yan, Qualitative properties of Hamiltonian difference systems, J. Math. Anal. Appl. 171 (1992) 334–345.

[15] R. Hilscher, Disconjugacy of symplectic systems and positivity of block tridiagonal matrices, Rocky Mountain J. Math. 29 (4) (1999) 1301–1319.

[16] R. Hilscher, V. Zeidan, Discrete optimal control: the accessory problem and necessary optimality conditions, J. Math. Anal. Appl. 243 (2) (2000) 429–452.

[17] R. Hilscher, V. Zeidan, Second order sufficiency criteria for a discrete optimal control problem, J. Differ. Equations Appl. 8 (6) (2002) 573–602.

[18] R. Hilscher, V. Zeidan, Nonnegativity of a discrete quadratic functional in terms of the (strengthened) Legendre and Jacobi conditions, in: R.P. Agarwal (Ed.), Advances in Difference Equations IV, Comput. Math. Appl., to appear.

[19] D.G. Luenberger, Linear and Nonlinear Programming, 2nd Edition, Addison-Wesley, Reading, MA, 1984.
[20] G. Stefani, P. Zezza, Constrained regular LQ-control problems, SIAM J. Control Optim. 35 (3) (1997) 876–900.
[21] V. Zeidan, Sufficiency criteria via focal points and via coupled points, SIAM J. Control Optim. 30 (1) (1992) 82–98.
[22] V. Zeidan, Sufficient conditions for variational problems with variable endpoints: coupled points, Appl. Math. Optim. 27 (2) (1993) 191–209.
[23] V. Zeidan, Nonnegativity and positivity of a quadratic functional, in: R.P. Agarwal, M. Bohner (Eds.), Discrete and Continuous Hamiltonian Systems, Dynam. Systems Appl. 8 (3–4) (1999) 571–588.
[24] V. Zeidan, P. Zezza, Coupled points in the calculus of variations and applications to periodic problems, Trans. Amer. Math. Soc. 315 (1) (1989) 323–335.
[25] P. Zezza, Jacobi condition for elliptic forms in Hilbert spaces, J. Optim. Theory Appl. 76 (2) (1993) 357–380.